Hans Christian Nordtveit Kvernenes, student 10.06.14
UiB, HF, FoF, Fil251 - Bacheloroppgave (7090 words)

Why negative probability can have an ontological status from a Quinean point of view and its application in probabilistic epistemic logic

Structure

Abstract

From a traditional point of view probabilities have only been in the interval from zero to one. Everything beyond or below that has been considered nonsense by most people, even though researchers within the fields of quantum mechanics and financial mathematics have argued for an extended interval for probabilities. This text will present examples for probability values that are beyond the traditional interval, and primarily an extension to negative values, and argue for its existence from a Quinean perspective. After that we will argue that negative probability can be used in epistemic logic, and especially Kooi's (2003) extension, probabilistic dynamic epistemic logic, PDEL. This system is based on Kolmogorov's axioms for probability and will therefore make probabilities with negative values undefined or wrong by definition. Burgin (2010) has made an extension of Kolmogorov's axioms to include negative values and proved it to be consistent. We will therefore see if it seems possible to use an alternative system that takes negative probability values in Kooi's logic instead of Kolmogorov's.

1. Introduction

1.1. General introduction

This text is separated in two main parts, one about metaphysics and one about epistemic logic. The metaphysical part will explain the ontology of negative probability values and why we should accept them from a Quinean perspective. The second part will use one example developed by Székely (2005) that he calls «half of a coin», and we will use this to see that it seems possible to use probabilistic epistemic logic to reason with probabilities, even when they are negative. We will however not define an extension or provide metalogical proofs such as proofs for completeness and soundness.

Probability can be defined as the measurement for uncertainty or certainty of events. We can assign a number to this certainty, normally from 0 to 1, where 1 is certain and 0 is certainly not. If we try to define it any closer there seems to be many different views on how we should define probability, and many of them requires metaphysical assumptions (Hájek 2011). Negative probability or negative probability values will be understood as events that have a certainty number that is lower than zero. Traditionally this is impossible by definition, but we want to explain why it does not seem to be necessary to have this restriction on probability values. The traditional view was presented by Kolmogorov and has been considered almost as a definition of probability theory (Hajek 2011).

Kolmogorov made three requirements or axioms that he claimed to be necessary restrictions on probability functions. The first one is non-negativity, no probability function can have a negative value. The second one is normalization, absolute certainty has probability value of 1. The third axiom is finite additivity, if two events are mutually exclusive the probability value of their union equals the sum of both probability values (Hájek 2011). We will discuss the validity of the first axiom in this text.

1.2. Ontology

The first part we will discuss the metaphysical aspect of negative probability to see if we can use it to describe reality in a satisfying way. We will show that there does not seem to be any contradiction by using it and that it can be useful in several different areas. Contradiction will be understood as two things that cannot be true at the same time of logical reasons. Contradiction implies inconsistency, that it is not rational to hold all parts as true. We will explain different concepts of probability and present a Quinean view to explain mathematics and their ontological status. This view claim that entities have an ontological status only if they are valuable or necessary to scientific theories (Hájek 2011). We will explain negative probability from this assumption. It can be discussed if this view implies a real ontological status or if it claims that ontological questions are not important, but in this text this does not seem to be a relevant question, and we will not discuss it further. Existence will then be understood as having relevance to scientific theories. That something is directly observable will be understood as something we can sense with our senses. For example numbers and probability does not seem to be directly observable. We only observe their instances.

1.3. Epistemic logic

In the second part we will describe Kooi's probabilistic dynamic epistemic logic, hereafter called PDEL. This is a logical system that is made for reasoning, thereby epistemic. It is also dynamic, meaning that it allows for changes or updates, and it is probabilistic, that it is possible to reason with measured uncertainty. It is an epistemic logic and it will therefore be grounded in agents and rational choices. An agent is a rational and active part of the game that can have different information that other agents in the game without leading to a contradiction. We will view agents as players in this text. To show that negative probability can be used in this system we will use half of a coin as an example and develop this into a game that we model. This object seems to yield negative probability values for some events. We will not produce any metalogical proves for negative probability, but merely show that it seems to be possible to extend this logic to take negative probability values.

2. Ontology of negative probability

2.1. What is probability?

2.1.1. What it is for something to have a probability value

To determine what it means for something to have negative probability and whether this exists or not we have to establish an understanding of what it would mean to say that something has a probability at all. Probability as most people recognizes it today is as a mathematical calculus based on Kolmogorov’s axioms for probability (Hájek 2011). This is an abstract mathematical theory that does not have a direct place in the observable world. We are not directly observing probabilities, only different events, but we are still using it in lot of different areas that are studying the nature, for example physics, social sciences and biology. When we apply the probability calculus in different areas we assume that the axioms provide a satisfying model of reality. It can be debated whether this calculus really is resembling this or that part of reality in a satisfying way, but in this text we assume that there exists parts of reality where probability axioms can resemble or at least be an adequate model. This assumption is based on induction, it seems to have been an adequate model so far, so we assume that it will continue to be.

2.1.2. Token probability and type probability

An important question to ask related to probability is: what do probability theory describe? The short answer to this question is that they describe the relation between different outcomes. It seems possible to separate between type probability and token probability. Type probabilities are probabilities where we speak about a set of objects or events, where probability values are distributed over a certain kind of objects, often with similar properties, for example with even six-sided dices. The type dice has a property of showing each side with probability 1/6, and we can say that when rolling a random dice from this set we have the probability 1/6 of getting side x. Token probability seems to be slightly different. They assign probability to specific events or cases and by doing so they appear different when the event has occurred (Briggs 2010, p. 1-2).

A person can claim on Monday that it is 1/6 chance that the dice he will throw on Tuesday will show 1. This will be a claim about a token event, throwing a dice on Tuesday. This seem to be a legitimate claim, but on wednesday we will know what side was shown on Tuesday, and the probability the we should sign for the event on Tuesday will be 1, as it is certain that the dice showed 1 or not. This claim may appear agent-relative, but that does not seem to be the case. We would never say that the probability for a dice that has already been thrown and shown to end at side x has a probability of 1/6 of showing side x. We should sign it to be 1. If we assume that the world is deterministic we see that probability assignments to token events seems to be a completely epistemic affair because the result has already been destined by the nature. A deterministic view seems therefore to refuse the existence of objective probability for token events.

Now we have described what entities probability theory describes, but we have not explained what we mean when we say that a certain event has some probability for happening.

2.2. Different concepts of probability

2.2.1. Three main concepts of probability

It seems to be possible to separate three main concepts of probability that we use. One concept is something we can call statistical probability. This means that we can have several tokens of a type see how many of the tokens have a certain property against how many have not. For example if we observe 10 cows and 7 of them are black we will get the statistical probability of 7/10 for a random chosen cow to be black. The main idea behind the statistical probability is that there exists properties in a system objectively, independently of us. In this interpretation probabilities are often made by induction. The thing that causes something to have a probability values is that it has happened before.

Things or events, tokens, does not have a probability assignment as a property in itself, but types can be assigned a probability value if several tokens of an event has happened before. Another concept is an evidential probability. This is a concept meaning there exists objective evidence that supports a certain statement. Evidence does not have to be connected to statistical probability. For example if you drive on the opposite side of the road, you will probably crash. This does not seem to directly connected to statistical probability as you do not seem to need the knowledge that x out of y people driving on the wrong side of the road crashes to understand or get the evidence. It is not connected to an agent's belief either, as the driver does not need to know that he will probably crash if he drives on the wrong side of the road. The probability for crashing seems to be independent of the agent (Hájek 2011). The last concept is epistemic probability. This is agent relative probability where different agents can have different probability assignments to the same proposition. It is an agents degree of certainty. For example that agent a think that it will probably be sunny tomorrow, while agentthinks it will probably be cloudy tomorrow. It seems to be possible to describe this kind through possible worlds. If an agent considers the probability for x to happen, we can describe it through a ratio between all possible worlds where x is happening and all possible worlds where it is not (Kooi 2003, p. 383-384).

2.2.2. Different variants of epistemic probability

It seems possible to separate epistemic probability even further. We can see that there can be an observational epistemic probability, meaning that an agent cannot decide which event he is observing, but still have a probability assignment for them. An example is that a player a watches another player *b* draw a card from the deck with two black cards and one red. a does not know what card has been drawn, but he can still assign a probability value to *b* drawing a black card. The other variant we can call bayesian epistemic probability because it is based on a bayesian approach. Bayesian epistemic probability is a probability assignment P = q by an agent a that event v will happen given another event h happening, $P_a(v \mid h) = q$. We can use the same example as with statistical probability, that an agent counts black cows, and from that give a probability assignment to a statement. The bayesian epistemic probability does not necessarily rely on a set or statistics though (Benthem 2009, p. 68-69) (Hájek 2011).

2.3. Ontology of negative and complex numbers from a naturalist point of view

2.3.1. The problem of the ontology of negative and complex numbers

As I see this the existence of negative probability values seem analogical to the existence of negative and complex numbers. A complex number is a number that is the sum of a real number a and the product of a real number b and the square root of −1, written i, a + bi (Priest 1998). If we assume that positive rationals have existence or references in the world that make them have an objective existence as describing properties of sets or set like entities negative numbers seem to have the same problem as negative probability values. Negative numbers does not seem to have any physical objects they refer to because all of our normal things does have a positive existence. It would be quite unclear what it would mean for something to have negative existence in world like that. We understand negative existence not as the same as negative facts, that things are not the case, for example that there is no water in the glass. This kind of negative facts does not seem to correspond with an objective negative existence, understood as existence that are opposite of positive, normal, existence. The same argument seems to follow for complex or imaginary numbers, as they do not seem to have objects or sets they directly refer to.

2.3.2. The naturalist assumption for mathematical theories

Then the important part of the question is: why should we accept negative and complex numbers in mathematics, even though they do not seem to have any objects to refer to? The answer to this seems to be quite debated and requires fundamental metaphysical assumptions. We will take a naturalist assumption based as developed by Quine. This naturalist assumption is that all things, both normal physical objects and mathematical entities, exists only by being parts of the best scientific theories (Hájek 2011) (Quine 1961 p. 44-45).

«Moreover, the abstract entities which are the substance of mathematics-ultimately classes and classes of classes and so un up-are another posit in the same spirit. Epistemologically these are myths on the same footing with physical objects and gods,

neither better nor worse except for differences in the degree to which they expedite our dealings with sense experiences.» (Quine 1961 p. 45)

The only thing that differs between things that have ontological status and those that have not is that they are valuable to predict our future. Scientific theories are the most efficient way to predict the future and entities exist if they are important to these scientific theories. Since mathematical entities are a very important part of todays scientific theories, for example Einsteins theory of relativity, mathematical entities exist. This is a variant of the indispensability argument that has been widely debated in the philosophy of mathematics (Colyvan 2014). From this we can also conclude that negative numbers exists in virtue of being useful or necessary in many scientific theories, such as economics and physics. We see that probability theory seems very useful in many theories, like quantum mechanics, game theory and mathematical economics, and therefore exists from a naturalist point of view. To explain why negative probability values exists we therefore have to explain how they can be applied in scientific theories and show that they not seem to be logically inconsistent nor inconsistent with mathematics or scientific theories. Scientific theories seem to be more fundamentally important than negative probability values and a consistency between them seems to be required for negative probabilities to have an ontological status at all (Hájek 2011).

2.4. Consistency of negative probability values

We have now established the requirements for us to show negative probabilities to exist, and one of them is to show that it is not inconsistent with more fundamental theories.

2.4.1. Logical consistency of negative probability

We will start by showing that there does not seem to be any logical contradiction by negative and positive probability values. We will assume that positive values will attach to positive events and negative values will attach to negative events (whatever this would happen to mean). We assume that an event cannot be both positive and negative at the same time. For an agent an event can have only one probability value for happening. They does not seem to logically contradict each other, as negative probability values will never attach to a positive event that could positively happen, and opposite. They appear to operate within different «classes» of events even though they can have an impact on each other, for example in a bayesian approach to probability theory.

2.4.2. Mathematical consistency of negative probability

Kolmogorov's first axiom for probability requires non-negativity for probability values (Hájek 2011). This is a direct contradiction to the very concept of negative probability values. To avoid this we have to assume that the axiom for non-negativity is either false or that it is only valid for a certain kind of the probabilities, the positive ones. Burgin (2010) has constructed a consistent extension to the traditional probability axioms. They are therefore consistent with each other and does not lead to a paradox in itself, but this does not really answer the question whether it is consistent with mathematics generally. We assume that it is, but it is very difficult to prove, as the mathematical system is very big.

2.4.3. Consistency between negative probability and scientific theories

Scientific theories are usually based on empirical data. It seems impossible to directly observe probability, neither positive nor negative, or numbers generally, even though some people may claim that we can observe the results of probability or instances of numbers or something like that. We cannot observe the nonexistence of something directly, and therefore not prove anything to be nonexistent by referring to empirically data. All this together cause that scientific theories will not be able to disprove negative probability values because its nonexistence seems to be epistemologically «hidden» for empirical studies. The only way to prove a nonexistence of such a thing would therefore be to prove that it is fundamentally, maybe logically, inconsistent in some or another way. It seems reasonable to conclude that negative probabilities are not inconsistent with scientific theories, neither today nor future empirical studies.

2.5. Usefulness of negative probability values in scientific theories

We assume that there is no inconsistency with more fundamental scientific theories, and our final step to establish the ontological status for negative probability values is to determine that negative probability values can be useful for scientific research. According to Quine entities are useful if they can be used to simplify or improve our possibility to predict the future based on past experiences. If it can be shown that negative probabilities are entities that will improve our capability to predict the future, we are commited to their existence (Quine 1961 p. 44-45). This can be accomplished by showing that negative probability can be necessary to provide answers or to show that it can work to simplify operations without being inconsistent or provide an inconsistent answer. The strategy we will use to show this will be to have a look at a theoretical object called half of a coin that could be useful in mathematical economics and possibly also decision theory, but we will also explain shortly that negative probability seems to be useful in quantum mechanics.

2.5.1. Usefulness in quantum mechanics

Many different physicists have argued for the use of negative probability values to explain processes on a quantum level. Paul Dirac (1942) argues for the existence of negative energy that can connect to negative probability.

«Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative sum of money, since the equations which express the important properties of energies and probabilities can still be used when they are negative. Thus negative energies and probabilities should be considered simply as things which do not appear in experimental results. » (Dirac 1942, p. 8)

He draws a parallel between negative probability and negative money (Dirac 1942, p. 8). We can see that Dirac emphasized not only the thought of negative probability, but also negative energy. The idea of negative energy seems to be relevant if we want to use negative probability in physics in the way Dirac suggested. In spite of their lack of direct reference in experiments he argue for their existence and usefulness in quantum mechanics (Dirac 1942).

Another person arguing for negative probability in quantum mechanics is Richard Feynman (1987). He argued mainly for its application in his main field, electrodynamics, but in the text he uses other examples from quantum mechanics to illustrate his view. The arguments he is using is mainly based on the analogy to negative numbers and its practical use. He is using many different examples about two-state systems where he claims that negative probabilities can be used. These arguments seem to correspond to our naturalist assumption, that the important part is to determine whether it is useful for scientific theories (Feynman 1987). We will not go further into quantum mechanics in this text, but merely accept that negative probability values can be useful to describe quantum mechanics.

2.5.2. Usefulness in economics

Haug (2004) argues for the use of negative probability in mathematical finance. He is using a concrete example from finance that involves pricing where he derives negative probability values in certain cases. There are however no new problem that is being solved, but he argues that negative probabilities can make calculations easier and more efficient and can possibly solve other problems that have not been solved before, if we examine it more thorough (Haug 2004, p. 35-37). There are many references to Dirac, Feynman and other physicists in his text and he claims that hidden, not directly observable, variables in finance may be modeled with negative probabilities as they do it in quantum mechanics. This shows us however that negative probability values can be useful in finance pricing and modeling.

Székely (2005) introduced a concept he calls half of a coin. We will present this object more closely later in this text. According to Székely also presents negative probability as corresponding to negative payments in economics. A negative payment could be a withdrawal from a bank account. He uses a similarly understanding of negative probability as Dirac, but also claims that it could play an important role in financial modeling (Székely 2005).

The object, half of coin, he introduces is an instance of a more general theorem stating that all signed probability distributions there exists two probability functions such that the product of the functions equals the original probability distribution (Székely 2005, p. 67). In many cases these functions will produce a negative probability for some events. The operation can therefore be done with all kinds of normal objects with normal probability distribution such as n-sided dices causing them to give a negative value to certain events. It can from here be argued that this can be shown to be useful in financial modeling if you want to model for example fractions or interest (Szekely 2005, p. 68).

3. Negative probability in epistemic logic

3.1. Kooi's probabilistic dynamic epistemic logic

3.1.1. Probabilistic epistemic model

With a countable set of propositional variables $\mathcal{P}$ and a finite set of agents $\mathcal{A}$, a probabilistic epistemic model is $M = (W, R, V, P)$

1. $W \neq \emptyset$; Set of possible worlds or conditions
2. R : Set of accessibility relations R^a for each agent $a \in \mathcal{A}$
3. V : For each $v \in \mathcal{P}$ v gets assigned a set $w \in W$ such that v is true in all w
4. P : All $a \in \mathcal{A}$ gets assigned a probability function for each $w \in W$.

3.1.2. Purely probabilistic model

With a finite set of agents $\mathcal{A}$ and a non-empty set *E,* a purely probabilistic model is M = (W, R, P)

1. W ≠ Ø; Set of possible worlds or conditions
2. R : Set of accessibility relations R^a for each agent $a \in \mathcal{A}$
3. P : All w ∈ W gets assigned a probability function

3.1.3. Syntax of PDEL

Let a countable set of propositional variables $\mathcal{P}$ and a finite set of agents $\mathcal{A}$ be given. The language PDEL is given by following rule in extended Backus-Naur form:

$$\phi ::= p \mid \neg\phi \mid \phi\wedge\psi \mid \Box_\alpha\phi \mid [\phi]\psi \mid q_1\mathbf{P}_a(\phi_1) + ... + q_n\mathbf{P}_a(\phi_n) \geq q$$

Where $p \in \mathcal{P}$, $a \in \mathcal{A}$ and $q_1, ..., q_k$ and q are rational numbers.

$\sum^{n}_{i=1} q_i\mathbf{P}_a(\phi_i) \geq q$: $q_1\mathbf{P}_a(\phi_1) + ... + q_n\mathbf{P}_a(\phi_n) \geq q$

$q_1\mathbf{P}_a(\phi) \geq q_2\mathbf{P}_a(\phi)$: $q_1\mathbf{P}_a(\phi) - q_2\mathbf{P}_a(\phi) \geq 0$

$\sum^{n}_{i=1} q_i \mathbf{P}_a(\phi_i) \leq q$: $\sum^{n}_{i=1} -q_i \mathbf{P}_a(\phi_i) \geq -q$

$\sum^{n}_{i=1} q_i \mathbf{P}_a(\phi_i) < q$: $\neg(\sum^{n}_{i=1} q_i \mathbf{P}_a(\phi_i) \geq q)$

$\sum^{n}_{i=1} q_i \mathbf{P}_a(\phi_i) > q$: $\neg(\sum^{n}_{i=1} q_i \mathbf{P}_a(\phi_i) \leq q)$

$\sum^{n}_{i=1} q_i \mathbf{P}_a(\phi_i) = q$: $(\sum^{n}_{i=1} q_i \mathbf{P}_a(\phi_i) \leq q) \wedge (\sum^{n}_{i=1} q_i \mathbf{P}_a(\phi_i) \geq q)$

(Kooi 2003, s. 387)

The syntax seems to not cause any problems for negative probability values. All abbreviations are valid concepts with negative values.

3.1.4. Semantics for PDEL

$M, w \vDash p$ iff $w \in V(p)$

$M, w \vDash \neg\phi$ iff $M, w \nvDash \phi$

$M, w \vDash \phi \wedge \psi$ iff $M, w \vDash \phi$ and M, s $\vDash \psi$

$M, w \vDash \Box_\alpha \phi$ iff for all $w' \in W$: if $wR^a w'$, then $M, w' \vDash \phi$

$M, w \vDash [\phi]\psi$ iff $(M_\phi, w_\phi) \vDash \psi$

$M, w \vDash \sum^{n}_{i=1} q_i \mathbf{P}_a(\phi_i) \geq q$ iff $\sum^{n}_{i=1} q_i \mathrm{P}(a, w)(\phi_i) \geq q$

where $P(a, w)(\phi_i) = P(a, w)(\{v \in dom(P(a, w)) \mid (M, v) \vDash \phi\})$

3.1.5. Semantics for updates

With a probabilistic epistemic model $M = (W, R, V, P)$ and a world $w \in W$ an updated model is $M_\phi = (W_\phi, R_\phi, V_\phi, P_\phi)$.

W_ϕ = W

R_ϕ^a = $\{(u, v) \mid (u, v) \in R(a) \text{ and } (M, v) \vDash \phi\}$

V_ϕ = V

$dom(P_\phi(a, u))$ = $dom(P(a, u))$ if $P(a, u)(\phi) = 0$

$\{v \in dom(P(a, u)) \mid (M, v) \vDash \phi\}$ otherwise

$$P_\phi(a, u)(v) = \begin{cases} P(a, u)(v) & \text{if } P(a, u)(\phi) = 0 \\ \dfrac{P(a, u)(v)}{P(a, u)(\phi)} & \text{otherwise} \end{cases}$$

3.1.6. Multiplication

Let an epistemic model *M* = (*W, R, V)* and a purely epistemic model M = (W, R, P) be given. A multiplied model $\mathfrak{M} = (\mathfrak{W}, \mathfrak{R}, \mathfrak{V}, \mathfrak{P})$ is $M \otimes$ M.

$$\mathfrak{W} = W \otimes \mathrm{W}$$

$$\mathfrak{R}(a) = \{((w, \mathrm{w}), (v, \mathrm{v})) \mid wR^a v \wedge \mathrm{wR}^a\mathrm{v}\}$$

$$\mathfrak{V}(p) = \mathrm{V}(p) \times \mathrm{W}$$

$$dom(\mathfrak{P}(a, (w, \mathrm{w}))) = \{v \mid wR^a v\} \times \{\mathrm{w}\}$$

$$\mathfrak{P}(a, (w, \mathrm{w}))(v, \mathrm{w}) = \frac{\mathrm{P(w)}(v)}{\sum_{(u,\, \mathrm{w}) \in dom(\mathfrak{P}(a, (w, \mathrm{w})))} \mathrm{P(w)}(u)}$$

(Kooi 2003, p. 387-388, 392-393)

3.2. Half of a coin as an example of negative probability

Half of a coin is an instance from the theorem he stated. This is a coin that has infinitely many sides numbered from 0 and if you throw two of them their probability add up the probability distribution for a normal coin with two sides, 0.5 for side 0 and 0.5 for side 1. The probability for one coin to land on one side is (1+z)/2, and the probability for two coins to land on one side is $((1+z)/2)^2$. Székely defines half of a coin as $\sqrt{(1+z/2)}$. He derives it to this following formula $p_n = (-1)^{n-1}\sqrt{(2(C_{n-1}/4^n))}$ n = 0, 1, ... where n is n side of the coin and C_n is a catalan number. Whenever n is an odd number the probability value for n p_n is a positive number, but whenever n is an even number p_n is a negative number (Székely 2005, p. 66-67).

This strange object does not seem to be a contradiction or inconsistent, and it will therefore not seem fundamentally wrong to use this as an example. We will use this coin to build a game and a model in PDEL.

3.3. Probabilistic epistemic model of half of a coin

3.3.1. Formal model of half of a coin

Probabilistic epistemic model for half of a coin and an agent a with full accessibility. s_n means that the coin is showing side n.

$\mathscr{A} = \{ a \}$

$\mathcal{P} = \{ s_1, s_2, \dots s_\infty \}$

$W = \{ w_1, w_2, \dots w_\infty \}$

R : $w_1 R^a w_1, w_1 R^a w_2, \dots w_1 R^a w_n; w_2 R^a w_1, w_1 R^a w_2, \dots w_2 R^a w_n; w_m R^a w_1, w_m R^a w_2, \dots w_m R^a w_n$

V : $V(s_1, w_1)=1, V(s_1, w_2)=0, \dots V(s_1, w_n)=0; V(s_2, w_2)=1, V(s_1, w_1)=0, \dots V(s_2, w_n)=0; V(s_n, w_n)=1, V(s_n, w_m)=0, \dots V(s_n, w_m)=0$

P : $P_a(w_1) = (-1)^{1-1}\sqrt{(2(C_{1-1}/4^1))}, P_a(w_2) = (-1)^{2-1}\sqrt{(2(C_{2-1}/4^2))}, \dots P_a(w_n) = (-1)^{n-1}\sqrt{(2(C_{n-1}/4^n))}$

This is a probabilistic epistemic model for an agent a that have full access to the «game». All worlds are accessible for a. The propositional variables are different situations or sides the coin has. All the possible worlds are accessible for a. Since if one side is the case it prevents the other sides from being the case we assign one and only one possible world to each propositional variable and the truth value for variable s_n in world w_n will be true, and false for all m that is not n. The probability for the world n to be the case is given by instance n of the formula that is describing the probability assignment of half of a coin.

There does not seem to be any problems by modeling half of a coin in a formal model.

3.3.2. Instances of half of a coin's probability function from zero to six

Instances of $p_n = (-1)^{n-1}\sqrt{(2(C_{n-1}/4^n))}$, where n = 0, 1, 2, 3, 4, 5, 6

$n = 0 : (-1)^{0-1}\sqrt{(2(C_{0-1}/4^0))} = -i$

$n = 1 : (-1)^{1-1}\sqrt{(2(C_{1-1}/4^1))} = 1/\sqrt{2} \approx 0.7$

$n = 2 : (-1)^{2-1}\sqrt{(2(C_{2-1}/4^2))} = -(1/(2\sqrt{2})) \approx -0.35$

$n = 3 : (-1)^{3-1}\sqrt{(2(C_{3-1}/4^3))} = 1/4 = 0,25$

$n = 4 : (-1)^{4-1}\sqrt{(2(C_{4-1}/4^4))} = -(\sqrt{(5/2)}/8) \approx -0.2$

$n = 5 : (-1)^{5-1}\sqrt{(2(C_{5-1}/4^5))} = \sqrt{(7)}/16 \approx 0.17$

$n = 6 : (-1)^{6-1}\sqrt{(2(C_{6-1}/4^6))} = -(\sqrt{(21)}/32) \approx -0.14$

These calculations are done to make it easier to understand the next part, as we will model a game with both half of a coin and a dice. Székely claims that $C_{0-1} = C_{-1}$ can be defined as –1/2 (Székely 2005, p. 67).

3.4. Model of a game with half of a coin and a dice

3.4.1. Rules for the game

The example is based on the example found in Kooi's article (2003, p. 401), but it is modified. An agent b tosses half of a coin or a normal and even dice with six sides. b knows which one he is tossing, but a does not know this. a know that b tosses either a half of a coin or an even dice. a is then offered to bet whether the result is 4 or not, ¬4.

The probability for the result to be 4 with the dice is 1/6, and 5/6 for it to not be the case ¬4. The probability for the result to be 4 with the half of a coin is $(-1)^{4-1}\sqrt{(2(C_{4-1}/4^4))} = -(\sqrt{(5/2)}/8) \approx -0.2$ and $1-(-(\sqrt{(5/2)}/8)) \approx 1.2$ for it to not be the case.

Both of the players are going to guess. If a player is guessing correctly and opposite of the other player he receives 90 euros and the other player nothing. If they both guess the same they both get 12 euros each. If this is a game with a normal dice it would be an advance for a to guess the opposite of b. b will rationally guess ¬4 because b's temporary expected payoff is 75 euros, 90*(5/6) = 75. If a guesses the opposite of b a's expected payoff 15 euros, 15 euro, 90*(1/6) = 15, but if a guesses the same as *b* *a*'s payoff will be 12 euros. If the game is with a half of a coin *b*'s temporary expected payoff will be around 108 euros if he guesses ¬4, 90*(1−(−(√(5/2)/8))) = 90(1+(√(2/5)/8)) ≈ 108, and around −18 euros if he is guessing, 90*(−(√(5/2)/8)) = −((45√(5/2))/4) ≈ −18. *b* will then rationally guess ¬4, and *a*'s expected payoff by guessing opposite of *b* will be around −18 euros. By guessing the same as *b* *a*'s payoff will be 12 euros. By the game with a half of a coin *b*'s expected payoff will change to 12 euros because it is expected for *a* to guess the same as b.

At first sight this calculation of expected payoff with a half of a coin seems contra intuitive because *b*'s expected payoff by guessing ¬4 is more than the amount they are playing about. This is also strange that expected payoff sometimes happens to be negative, but this is connected to the other ones strange expected payoff. The reason for these strange payoffs is the structure of this strange object that seems very different from objects we see in our daily life. Whenever a payoff is more than the sum being played with there is another payoff that causes the total sum of the payoffs to equal to the sum being played with. 90(1+(√(2/5)/8))−(−((45√(5/2))/4)) = 90. We can explain this situation more naturally by claiming that this causes the loosing player to give the winning player money.

3.4.2. Formal model of the game

We can now make a probabilistic epistemic model of this game. t_n means that the dice will show side n, and s_n means that the coin will show side n. We connect each possible proposition to one possible world.

$\mathscr{A} = \{ a, b \}$

$\mathcal{P} = \{ t_1, t_2, t_3, t_4, t_5, t_6, s_1, s_2, \dots s_\infty \}$

$W = \{ u_1, u_2, u_3, u_4, u_5, u_6, w_1, w_2, \dots w_\infty \}$

R : $w_1R^aw_1, w_1R^aw_2, \ldots w_1R^aw_n; w_2R^aw_1, w_1R^aw_2, \ldots w_2R^aw_n; w_mR^aw_1, w_mR^aw_2, \ldots w_mR^aw_n,$

$w_1R^bw_1, w_1R^bw_2, \ldots w_1R^bw_n; w_2R^bw_1, w_1R^bw_2, \ldots w_2R^bw_n; w_mR^bw_1, w_mR^bw_2, \ldots w_mR^bw_n$

$u_1R^au_1, u_1R^au_2, u_1R^au_3, u_1R^au_4, u_1R^au_5, u_1R^au_6,$

$u_1R^bu_1, u_1R^bu_2, u_1R^bu_3, u_1R^bu_4, u_1R^bu_5, u_1R^bu_6,$

$u_1R^aw_1, u_1R^aw_2, \ldots u_1R^aw_n; u_2R^aw_1, u_2R^aw_2, \ldots u_2R^aw_n; u_3R^aw_1, u_3R^aw_2, \ldots u_3R^aw_n; u_4R^aw_1,$

$u_4R^aw_2, \ldots u_4R^aw_n; u_5R^aw_1, u_5R^aw_2, \ldots u_5R^aw_n; u_6R^aw_1, u_6R^aw_2, \ldots u_6R^aw_n; w_1R^au_1,$

$w_2R^au_1, \ldots w_nR^au_1; w_1R^au_2, w_2R^au_2, \ldots w_nR^au_2; w_1R^au_3, w_2R^au_3, \ldots w_nR^au_3; w_1R^au_4,$

$w_2R^au_4, \ldots w_nR^au_4; w_1R^au_5, w_2R^au_5, \ldots w_nR^au_5; w_1R^au_6, w_2R^au_6, \ldots w_nR^au_6;$

V : $V(t_1, u_1)=1, V(t_2, u_2)=1, V(t_3, u_3)=1, V(t_4, u_4)=1, V(t_5, u_5)=1, V(t_6, u_6)=1; V(s_1, w_1)=1, V(s_1, w_2)=0, \ldots V(s_1, w_n)=0; V(s_2, w_2)=1, V(s_1, w_1)=0, \ldots V(s_2, w_n)=0; V(s_n, w_n)=1, V(s_n, w_m)=0, \ldots V(s_n, w_m)=0$

P : $P_a(u_1) = 1/6, P_a(u_2) = 1/6, P_a(u_3) = 1/6, P_a(u_4) = 1/6, P_a(u_5) = 1/6, P_a(u_6) = 1/6,$

$P_b(u_1) = 1/6, P_b(u_2) = 1/6, P_b(u_3) = 1/6, P_b(u_4) = 1/6, P_b(u_5) = 1/6, P_b(u_6) = 1/6,$

$P_a(w_1) = (-1)^{1-1}\sqrt{(2(C_{1-1}/4^1))}, P_a(w_2) = (-1)^{2-1}\sqrt{(2(C_{2-1}/4^2))}, \ldots P_a(w_n) = (-1)^{n-1}\sqrt{(2(C_{n-1}/4^n))},$

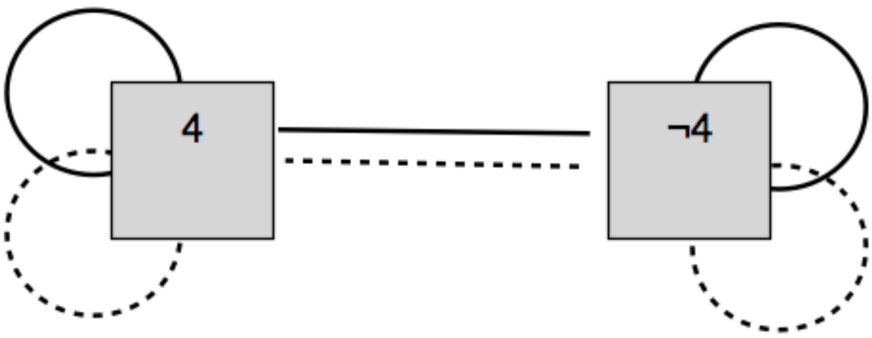
$P_b(w_1) = (-1)^{1-1}\sqrt{(2(C_{1-1}/4^1))}, P_b(w_2) = (-1)^{2-1}\sqrt{(2(C_{2-1}/4^2))}, \ldots P_b(w_n) = (-1)^{n-1}\sqrt{(2(C_{n-1}/4^n))}$

This is an extended version of the model with only one agent and half of a coin. All worlds are accessible for both agents. A dice land on one side and only one side if it is played with. The same is the case with half of a coin. The truth value for t_n is true in u_n and only in u_n. The truth value for s_n is true in w_n and only w_n. The probability for u_n to be true is 1/6, where n is 1, 2, 3, 4, 5, 6. The probability for w_n to be true is $(-1)^{n-1}\sqrt{(2(C_{n-1}/4^n))}$, where n is 0, 1, ... ∞.

3.4.3. Multiplied model

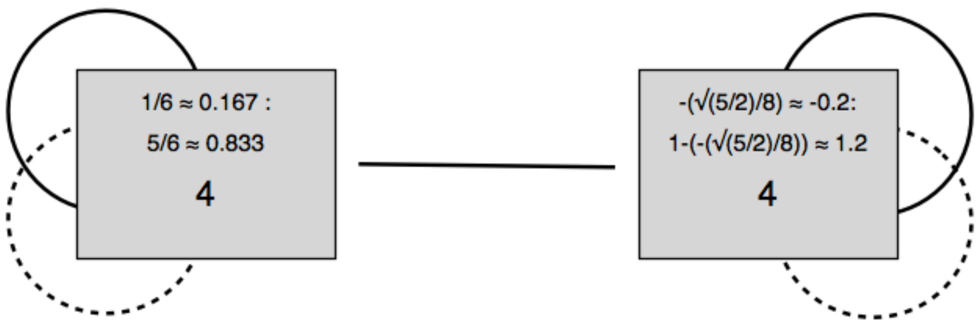
We can now make an epistemic model and a purely probabilistic model to get a multiplied model showing the whole game. Solid lines are accessibility relations for agent *a*. Dashed lines are accessibility relations for agent *b*. The upper dashed box in the multiplied model is the game with a dice and the lower is for half of a coin.

Epistemic model:



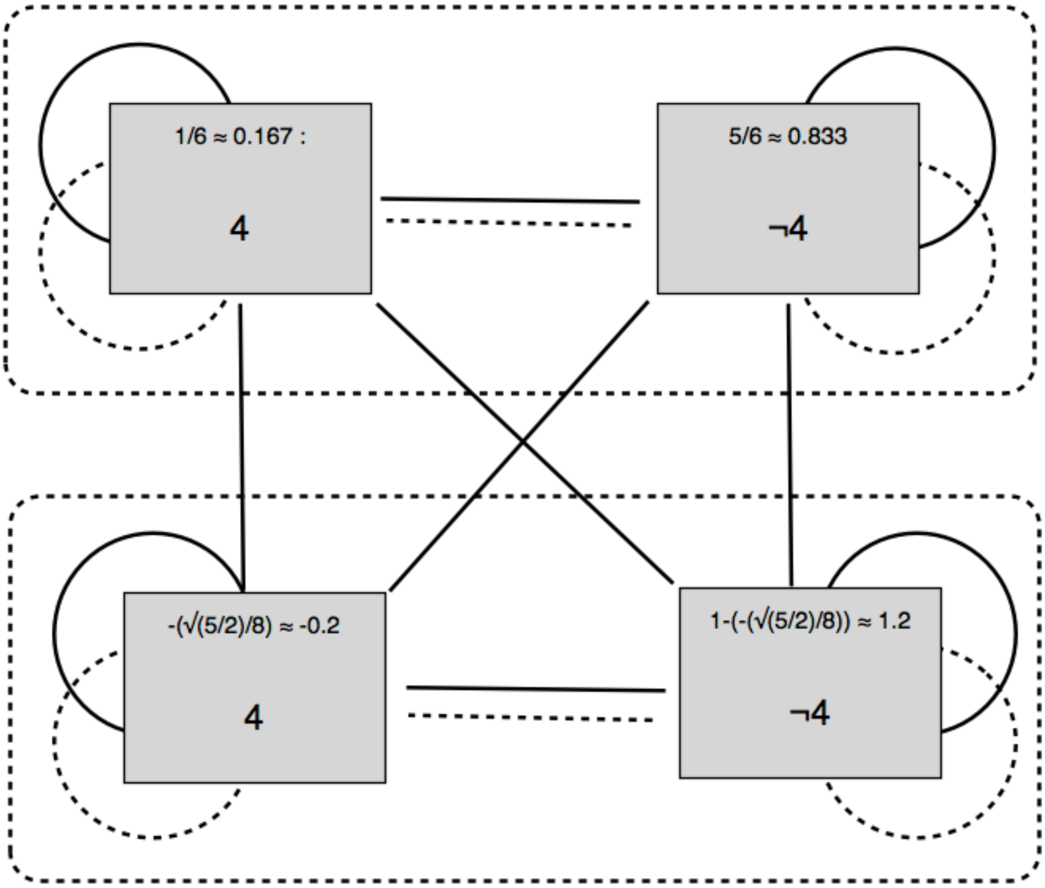
This shows that both of the agents cannot know epistemically whether 4 will be the case or not.

Purely probabilistic model:



This model shows that agent *a* cannot differ between the game with a dice or game with half of a coin. Agent *b* can differ. Probability distributions are placed in the boxes to show the different games.

Multiplied model:



This model is the combined version of the two previous models. We have used the rule for multiplication to make this model. It shows the whole game with probability distributions. They are both accessible from each other for a, but not for b. b knows what game is played.

3.4.4. Remarks on the game

Agent a is not able to determine what he should do in this game because he has no way of differ between the game with a dice and the game with half of a coin. a needs information, ether epistemic certainty or a probability distribution, about the different games he is playing to determine the best strategy. This is higher order information, but if a gets this information he does not have problems of deciding a strategy. If the game is with a dice a should guess the opposite of b, 4, but if the game is with half of a coin a should guess ¬4, the same as b.

It does not seem to be any practical problems by modeling games where negative probability values are used. We can model them the same way as games with only positive values. There does not seem to be any problems by modeling games with probability values higher than 1 either. We have however not explored this possibility thorough in this text.

4. Conclusion

It seems possible to separate probabilities to token and type probabilities, and describe the reality in terms of these. We can differ between several concepts of probability, where epistemic probability is one of those concepts. The question if negative probabilities exist seems to be parallel to whether negative and complex numbers exists generally. Mathematics seem to exist in virtue of being useful or necessary to scientific theories, and negative probabilities could be an extension.

Negative probability values seem to be useful and are not inconsistent in different scientific areas. From a Quinean perspective negative probability therefore have an ontological status. We have showed that negative probabilities can be used in finance, and that several physicists argue for using them in quantum mechanics too.

Kooi developed an probabilistic dynamic epistemic logic, PDEL, where negative probability values seem to be a natural extension. There does not seem to be any problems for the language to take these values. Since we have established the ontological status and usefulness of negative probabilities it seems useful to see if it is possible to incorporate them in logical systems so that they can be systematically reasoned with. When negative probabilities are used in economics, it seems natural to establish it in an epistemic logic that can describe agents in for example economical games.

There does not seem to be any problem to model a game with negative probabilities in PDEL. We used half of a coin to establish the negative values. The game can be modeled in the same way as games with normal probability values.

It seems to be useful to establish an axiomatic extension of PDEL with a proof system to accept negative probability values and see if it is possible to give proof for its soundness and completeness with the extension. An extension like this cannot directly build on Kolmogorov's axioms, but it has been proposed other systems that takes negative values into account. A epistemic system with this axiomatic basis will possibly be a powerful system to model games where negative probabilities are involved. It also seems to be interesting to establish the ontology and metaphysical implications of negative events, as they could be strictly connected to negative probabilities.

5. References

Benthem, J., J. Gerbrandy, B. Kooi, 2008, *Dynamic Update with Probabilites* [Internet], URL: < http://dare.uva.nl/document/170086 >, downloaded 5. June 2014

Briggs, R., 2010, *The Metaphysics of Chance* [Internet], URL: < http://onlinelibrary.wiley.com/store/10.1111/j.1747-9991.2010.00345.x/asset/j.1747-9991.2010.00345.x.pdf?v=1&t=hw217ps9&s=6d8e8080464386393fbe85e7e64bd2cb70a4724b >, downloaded 5. June

Burgin, M., 2010, *Interpretations of Negative Probabilities* [Internet], URL: < http://arxiv.org/pdf/1008.1287.pdf >, downloaded 5. June 2014

Colyvan, M., 2014, *Indispensability Arguments in Philosophy of Mathematics* [Internet], URL: < http://plato.stanford.edu/entries/mathphil-indis/ >, downloaded 6. June 2014

Dirac, P. A. M., 1942, *Bakerian Lecture. The Physical Interpretation of Quantum Mechanics* [Internet], URL: < http://rspa.royalsocietypublishing.org/content/180/980/1.full.pdf+html >, downloaded 5. June 2014

Feynman, R. P., 1987, *Negative Probability* [Internet], URL: < http://cds.cern.ch/record/154856/files/pre-27827.pdf >, downloaded 5. June 2014

Hájek, A., 2011, *Interpretations of Probability* [Internet], URL: < http://plato.stanford.edu/entries/probability-interpret/ >, downloaded 5. June 2014

Haug, E. G., 2004, *Why so Negative to Negative Probabilities?* [Internet], URL: < http://www.espenhaug.com/NegativeProbabilitiesHaug.pdf >, downloaded 5. June 2014

Kooi, B. P., 2003, *Probabilistic Dynamic Epistemic Logic* [Internet], URL: < http://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.99.4621&rep=rep1&type=pdf >, downloaded 5. June 2014

Priest, G., 1998, Numbers. In E. Craig (Ed.), Routledge Encyclopedia of Philosophy. London: Routledge, from http://www.rep.routledge.com/article/Y026SECT2, downloaded June 05, 2014

Quine, W., 1961, «Two Dogmas of Empiricism» in *From a Logical Point of View,* Harper Torchbooks, s. 20-46, Copy by University of Bergen

Székely, G. J., 2005, *Half of a Coin: Negative Probabilities* [Internet], URL: < http://www.wilmott.com/pdfs/100609_gjs.pdf >, downloaded 5. June 2014